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How to enrich description logics with fuzziness

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***Abstract*—The paper describes the relation between fuzzy and non-fuzzy description logics. While the process of transformation from a description logic to a fuzzy logic is a trivial inclusion, the other way of reducing information from fuzzy logic to description logic is a difficult task. The paper gives an overview about current research in these areas and describes the difference between tasks for description logics and fuzzy logics.**

Keywords*—*Semantic Web; Fuzzy Description Logic; Uncertainty

I. INTRODUCTION

Formal systems of logic are used to represent and store the knowledge of the world systematically. In some domains, knowledge is not certain in every case and ambiguity could offer space for interpretation. This necessitates the management of uncertainty. Description logics (DL) can handle inference and integrity rules to let knowledge grow and to keep consistency, but they don't deal with uncertainty.

In systems without management of uncertainty, an axiom is either true or false (or unknown). In systems with it, numeric values can be attached to axioms, that describe the certainty of the axiom. In such systems with closed world assumption, the numeric value expresses the probability of the axiom to be true or false respectively (probabilistic logic). In systems with open world assumption it describes the possibility of the axiom to be true or the necessity of it not to be false (possibilistic logic). This paper describes the extension of existing knowledge bases with the management of uncertainty. It explains the enrichment of description logics and gives an example of the whole process. This process can be divided in several tasks, as shown in figure I.

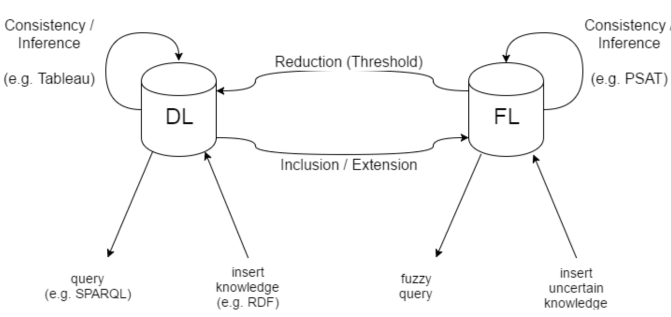

Fig. 1. processes from description logics to fuzzy logics

A description logic (DL) manages non-fuzzy knowledge. The database contains information and offers the possibility to add information, e.g. in RDF or Turtle format. There exist several algorithms to check the consistency of the inserted knowledge or to derive new knowledge by the use of inference rules. A user that requests for information, can query the database to extract some knowledge. For fuzzy logics (FL), these tasks are almost the same. One can insert knowledge or query and the system itself should be able to do inference and check consistency. To exchange knowledge between a description logic and a fuzzy logic, there are three possibilities. Since a fuzzy logic is more expressive than a description logic, one can include the knowledge that is stored within a description logic into a fuzzy system. On the other hand, since it is not possible to store uncertain information in a description logic, one can try to derive this information from the current knowledge and add it to the fuzzy logic, this is an extension of the existing knowledge. In order to go back to a description logic from a fuzzy logic, one can remove the uncertainty and include everything with a certainty upon a certain threshold.

This paper is organized in the following way. Section II is a background section and gives an introduction about the state of the art in the management of uncertainty and vagueness. The section discusses the most important aspects and theories in the area of uncertainty in the semantic web. Section III gives a short introduction to description logics and section IV to fuzzy logics. Then section V describes the process to get from one logic to another. Finally section VI gives a brief summary of the paper and gives an outlook.

II. RELATED WORK

One of the first occurrences of probabilistic logics is [13], which is a straight forward extension of propositional logics and is used widely as the basis for any kind of fuzzy logic. Algorithms that check the consistency of these Nilsson-style logics exist in different alignments, e.g. [5], [6], [7] or [14]. The bad news is the general PSAT algorithm that checks consistency of fuzzy propositional logics is NP-hard and not practical for knowledge bases with at least several hundreds of statements. Therefore, the approaches try to improve the speed of the algorithms and thereby sometimes loose accuracy.
On the other hand, for practical uses, the consistency in fuzzy databases is not very important, because the knowledge is imprecise or probabilistic anyhow. It is more important to get results for certain requests ranked by importance. To achieve this, one has to develop heuristic algorithms that don't work exactly correct, but lead to good results for practical purposes,

e.g. [5] or [15]. This is the case not only for queries, but also for reasoning, e.g. [3].

There are two conceptual different kinds of interpretations in fuzzy logics. Knowledge could be uncertain or vague. In uncertainty theory, an information is either true or false, but it is unknown. In imprecision or vagueness theory, an information is well known, but one would rather like to state a degree to emphasis if an information is completely fulfilled or only a little. The statement "it rains 10%" for example would state a light rain in vagueness theory. But it would state, that it rains in only 1 of 10 cases, in uncertainty theory. See [2] for further details.

In both cases, one could use different kinds of modeling. Probabilistic logic attaches a value in $[0, 1]$ to each information. This value is interpreted as a probability in uncertainty theory and is interpreted as a degree of emphasis of the information in imprecision or vagueness theory. Possibilistic logic attaches a range, i.e. two values in $[0, 1]$, to each information. The lower value is called necessity and the upper value is called possibility. Fuzzy logic attaches a probability distribution to each information, i.e. a function $P : [0, 1] \to [0, 1]$ with $\int_0^1 p(x)dx = 1$. To handle it easier, sometimes it is only allowed to take one of a small set of functions with customizable parameters.

III. Description Logic

This section is a short introduction to description logics. To keep things simple, only the base description logic $\mathcal{ALC}$ (attributive language with complement) is introduced. For further details, see [17].

A description logic consists of three arbitrary sets for naming roles, concepts and individuals. The set of role names is denoted by N_R, the set of concept names by N_C and the set of named individuals is denoted by N_O. In this case, the set of role names N_R is equal to the set of roles $\mathbf{R} = N_R$. But the set of concepts is more than only the names, it also contains several combinations of concepts.

Definition 1: Let N_C be a set of concept names and let $\mathbf{R}$ be a set of roles. The set of concepts $\mathbf{C}$ is the set, that contains the concept names, i.e. $N_C \subseteq \mathbf{C}$, and fulfills the following conditions.

$$C \in N_C \implies C \in \mathbf{C} \tag{1}$$
$$\top \in \mathbf{C} \tag{2}$$
$$\bot \in \mathbf{C} \tag{3}$$
$$A \subseteq N_O \implies A \in \mathbf{C} \tag{4}$$
$$C \in \mathbf{C} \implies \neg C \in \mathbf{C} \tag{5}$$
$$C, D \in \mathbf{C} \implies C \sqcap D \in \mathbf{C} \tag{6}$$
$$C, D \in \mathbf{C} \implies C \sqcup D \in \mathbf{C} \tag{7}$$
$$r \in \mathbf{R}, C \in \mathbf{C} \implies \exists r.C \in \mathbf{C} \tag{8}$$
$$r \in \mathbf{R}, C \in \mathbf{C} \implies \forall r.C \in \mathbf{C} \tag{9}$$

An element $C \in \mathbf{C}$ of this set is called concept.

The next definition describes the axioms (assertions) in a knowledge base. The axioms make statements about the relation between roles, concepts and individuals. In the case of $\mathcal{ALC}$, only TBox and ABox contain assertions.

Definition 2: Let N_A be a set of individual names, $\mathbf{C}$ be a set of concepts and $\mathbf{R}$ be a set of roles. An ABox is a finite set $\mathcal{A} \subset \{C(a) : a \in N_O, C \in \mathbf{C}\} \cup \{r(a, b) : a, b \in N_A, r \in \mathbf{R}\}$. A TBox is a finite set $\mathcal{T} \subset \{C \sqsubseteq D : C, D \in \mathbf{C}\}$. The elements of the ABox are called individual assertions, and the elements of the TBox are called General Concept Inclusions (GCI). A tuple that consists of a TBox and an ABox $(\mathcal{T}, \mathcal{A}) = \mathcal{K}$ is called a knowledge base.

The connection between the schema, described by the TBox, and the real world is the interpretation. An interpretation fills data into the boxes, the TBox in particular. Formally, it is a function $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ over the rule names, concept names and individual names. The basis of this is the set of all real world object which is called domain or universe of discourse and is denoted by $\Delta^{\mathcal{I}}$. The interpretation assigns each individual an element of the domain, each concept a subset of the domain and each role a subset of the Cartesian square of the domain. In addition, the interpretation function has to follow several rules.

Definition 3: An interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ over a signature (N_R, N_C, N_O) is a tuple, that consists of a domain $\Delta^{\mathcal{I}} \neq \emptyset$ and a function with the following constraints. An individual $a \in N_O$ must be mapped to an element of the domain, i.e. $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. A role $r \in \mathbf{R} = N_R$ must be mapped to a subset of the Cartesian square of the domain, i.e. $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. According to definition 1, a concept can occur in different ways. Aside from the fact that each concept $C \in \mathbf{C}$ must be mapped to a subset of the domain, i.e. $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, the composite concepts must fulfill the following rules, where $C, D \in \mathbf{C}$ are concepts, $r \in \mathbf{R}$ is a rule and $A = \{a_1, \ldots, a_n\} \subseteq N_O$ is a set of individuals.

$$\top^{\mathcal{I}} = \Delta^{\mathcal{I}} \tag{2}$$
$$\bot^{\mathcal{I}} = \emptyset \tag{3}$$
$$A^{\mathcal{I}} = \{a_1, \ldots, a_n\}^{\mathcal{I}} = \{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\} \tag{4}$$
$$(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \tag{5}$$
$$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}} \tag{6}$$
$$(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}} \tag{7}$$
$$(\exists r.C)^{\mathcal{I}} = \left\{\delta \in \Delta^{\mathcal{I}} : \exists \delta' \in C^{\mathcal{I}}, \langle \delta, \delta' \rangle \in r^{\mathcal{I}}\right\} \tag{8}$$
$$(\forall r.C)^{\mathcal{I}} = \left\{\delta \in \Delta^{\mathcal{I}} : \nexists \langle \delta, \delta' \rangle \in r^{\mathcal{I}}, \delta' \notin C^{\mathcal{I}}\right\} \tag{9}$$

Definition 4: In an interpretation $\mathcal{I}$ holds a TBox $\mathcal{T}$, denoted by $\mathcal{I} \models \mathcal{T}$, if and only if every GCI $(C \sqsubseteq D) \in \mathcal{T}$, where $C, D \in \mathbf{C}$ are concepts, fulfills $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

In an interpretation $\mathcal{I}$ holds an ABox $\mathcal{A}$, denoted by $\mathcal{I} \models \mathcal{A}$, if and only if every individual assertion $C(a) \in \mathcal{A}$, where $a \in N_O$ is an individual name and $C \in \mathbf{C}$ is a concept, fulfills $a^{\mathcal{I}} \in C^{\mathcal{I}}$ and every individual assertion $r(a, b) \in \mathcal{A}$, where $a, b \in N_O$ are individual names and $r \in \mathbf{R}$ is a role, fulfills $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in r^{\mathcal{I}}$.

In an interpretation $\mathcal{I}$ holds a knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, denoted by $\mathcal{I} \models \mathcal{K}$, if and only if $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$.

A knowledge base $\mathcal{K}$ is called consistent, if and only if there exists an interpretation $\mathcal{I}$, such that it holds the knowledge base, i.e. $\mathcal{I} \models \mathcal{K}$. It is called inconsistent otherwise.

If a knowledge base is inconsistent, i.e. it contains contradictions, it is not useful for inference, because one can infer anything.

The following knowledge base serves as an example to explain how concepts are computed. It contains the following individual names, concept names and role names, as shown in Example 16 of [17].

$$N_I = \{\texttt{zero}\}$$
$$N_C = \{\texttt{Prime}, \texttt{Positive}\}$$
$$N_R = \{\texttt{hasSuccessor}, \texttt{lessThan}, \texttt{multipleOf}\}$$

The domain is the set of all natural numbers including zero. Furthermore, the interpretation of individual names, concept names and role names is as follows.

$$\Delta^{\mathcal{I}} = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$$

$$\begin{aligned}
\texttt{zero}^{\mathcal{I}} &= 0 \\
\texttt{Prime}^{\mathcal{I}} &= \{n \in \mathbb{N} : k, l \in \mathbb{N}, n = k \cdot l \implies k = 1 \lor l = 1\} \\
\texttt{Positive}^{\mathcal{I}} &= \{1, 2, 3, \ldots\} \\
\texttt{hasSuccessor}^{\mathcal{I}} &= \{\langle n, n+1 \rangle : n \in \mathbb{N}_0\} \\
\texttt{lessThan}^{\mathcal{I}} &= \{\langle n, n' \rangle : n, n' \in \mathbb{N}_0, n < n'\} \\
\texttt{multipleOf}^{\mathcal{I}} &= \{\langle n, n' \rangle : n, n' \in \mathbb{N}_0, \exists k \in \mathbb{N} : n = k \cdot n'\}
\end{aligned}$$

The following exercise shows the composition of concepts and their results.

$\exists \mathtt{multipleOf}.\{\mathtt{zero}\}$

This expression is the concept of numbers, that are a multiple of zero, which is only zero itself.

$$\begin{aligned}
&(\exists \texttt{multipleOf}.\{\texttt{zero}\})^{\mathcal{I}} \\
&= \left\{\delta \in \Delta^{\mathcal{I}} : \exists \delta' \in \{\texttt{zero}\}^{\mathcal{I}}, \langle \delta, \delta' \rangle \in \texttt{multipleOf}^{\mathcal{I}}\right\} \\
&= \left\{\delta \in \Delta^{\mathcal{I}} : \langle \delta, 0 \rangle \in \texttt{multipleOf}^{\mathcal{I}}\right\} \\
&= \{n \in \mathbb{N}_0 : \exists k \in \mathbb{N} : n = k \cdot 0\} \\
&= \{0\}
\end{aligned}$$

The next exercise, from Exercise 2 in [17], demonstrates the computation of consistency. The expressions have to hold in $\mathcal{I}$. The expression would be called inconsistent or unsatisfiable otherwise.

$\mathtt{hasSuccessor} \sqsubseteq \mathtt{lessThan}$

This statement expresses that every two numbers that are connected by `hasSuccessor` are also connected by `lessThan`, i.e. any number is less than its successor. To prove consistency, one has to show that the left hand set is a subset of the right hand set.

$$\begin{aligned}
&\texttt{hasSuccessor}^{\mathcal{I}} \subseteq \texttt{lessThan}^{\mathcal{I}} \\
&\{\langle n, n+1 \rangle : n \in \mathbb{N}_0\} \subseteq \{\langle n, n' \rangle : n, n' \in \mathbb{N}_0, n < n'\}
\end{aligned}$$

So let $\langle \delta, \delta' \rangle \in \{\langle n, n+1 \rangle : n \in \mathbb{N}_0\}$ be an arbitrary element. There must be an $n \in \mathbb{N}_0$, such that $\delta = n$ and $\delta' = n + 1$. Since $\delta = n < n + 1 = \delta'$, $\langle \delta, \delta' \rangle \in \{\langle n, n' \rangle : n, n' \in \mathbb{N}_0, n < n'\}$.

IV. Fuzzy Logic

This section starts with an example that is extracted from [4] and then continues with a formal definition of fuzzy logics.

Three friends regularly go to a certain bar. The barkeeper states, that at least two of them are there every night. But each of the three friends states that he would only go to the bar in 60% of the nights.
Question: Is it consistent, i.e. does anybody lie or not?

In the above example, there are three friends $\Delta^{\mathcal{I}} = \{x_1, x_2, x_3\}$, and one concept $N_C = \{\text{Bar}\}$. There exist $2^3 = 8$ possible worlds in the non-fuzzy case:

$$\begin{aligned}
\text{Bar}^{\mathcal{I}} &= \emptyset \\
\text{Bar}^{\mathcal{I}} &= \{x_1\} \\
\text{Bar}^{\mathcal{I}} &= \{x_2\} \\
\text{Bar}^{\mathcal{I}} &= \{x_3\} \\
\text{Bar}^{\mathcal{I}} &= \{x_1, x_2\} \\
\text{Bar}^{\mathcal{I}} &= \{x_2, x_3\} \\
\text{Bar}^{\mathcal{I}} &= \{x_3, x_1\} \\
\text{Bar}^{\mathcal{I}} &= \Delta^{\mathcal{I}}
\end{aligned}$$

For the probabilistic case, one can extract 6 conditions from the example:

$$\begin{aligned}
\text{Bar}^{\bar{\mathcal{I}}}(x_1) &= 3/5 \\
\text{Bar}^{\bar{\mathcal{I}}}(x_2) &= 3/5 \\
\text{Bar}^{\bar{\mathcal{I}}}(x_3) &= 3/5 \\
\text{Bar}^{\bar{\mathcal{I}}}(x_1 \lor x_2) &= 1 \\
\text{Bar}^{\bar{\mathcal{I}}}(x_2 \lor x_3) &= 1 \\
\text{Bar}^{\bar{\mathcal{I}}}(x_3 \lor x_1) &= 1
\end{aligned}$$

The goal is to determine whether there exists a probability distribution over the 8 possible worlds that fulfills the 6 conditions. Formally: Is there a solution π to the following equations?

$$A\pi = p \quad \forall_{1 \le i \le 2^3} : \pi_i \ge 0 \quad \sum_{i=1}^{2^3} \pi_i = 1$$

Where $A \in \{0, 1\}^{6 \times 2^3}$ is a matrix that states the connection between the conditions and the possible worlds. $\pi \in [0, 1]^{(2^3)}$ is the probability distribution over the possible worlds, it is unknown what it looks like. The vector p contains the probabilities of the conditions. One can include the condition for the probability distribution to sum up to 1 in the equation system by adding a row of ones in the matrix and a 1 in the vector p, so that only $\pi \ge 0$ remains as an additional condition to the equation system. Then the equation system for the example of the friends in the bar is the following.

$$\begin{bmatrix}1&1&1&1&1&1&1&1\\0&1&0&0&1&0&1&1\\0&0&1&0&1&1&0&1\\0&0&0&1&0&1&1&1\\0&1&1&0&1&1&1&1\\0&0&1&1&1&1&1&1\\0&1&0&1&1&1&1&1\end{bmatrix}\cdot\begin{bmatrix}\pi_1\\\pi_2\\\pi_3\\\pi_4\\\pi_5\\\pi_6\\\pi_7\\\pi_8\end{bmatrix}=\begin{bmatrix}1\\3/5\\3/5\\3/5\\1\\1\\1\end{bmatrix}$$

This system has a set of solutions.

$$\pi=\begin{bmatrix}-c-1/5\\c+1/5\\c+1/5\\c+1/5\\-c+1/5\\-c+1/5\\-c+1/5\\c\end{bmatrix}$$

One can easily see that for every $c \in \mathbb{R}$ the condition $\pi \geq 0$ is not fulfilled. That means there is no solution, i.e. the knowledge base is inconsistent.

What if there are not three, but four friends, ceteris paribus, i.e. $\Delta^{\mathcal{I}} = \{x_1, x_2, x_3, x_4\}$? The amount of possible worlds is now $2^4 = 16$. Again, one has to determine the vector π. In this case it consists of the following possible worlds.

π_1	$\text{Bar}^{\mathcal{I}} = \emptyset$
π_2	$\text{Bar}^{\mathcal{I}} = \{x_1\}$
π_3	$\text{Bar}^{\mathcal{I}} = \{x_2\}$
π_4	$\text{Bar}^{\mathcal{I}} = \{x_3\}$
π_5	$\text{Bar}^{\mathcal{I}} = \{x_4\}$
π_6	$\text{Bar}^{\mathcal{I}} = \{x_1, x_2\}$
π_7	$\text{Bar}^{\mathcal{I}} = \{x_2, x_3\}$
π_8	$\text{Bar}^{\mathcal{I}} = \{x_3, x_4\}$
π_9	$\text{Bar}^{\mathcal{I}} = \{x_4, x_1\}$
π_{10}	$\text{Bar}^{\mathcal{I}} = \{x_1, x_3\}$
π_{11}	$\text{Bar}^{\mathcal{I}} = \{x_2, x_4\}$
π_{12}	$\text{Bar}^{\mathcal{I}} = \{x_1, x_2, x_3\}$
π_{13}	$\text{Bar}^{\mathcal{I}} = \{x_2, x_3, x_4\}$
π_{14}	$\text{Bar}^{\mathcal{I}} = \{x_3, x_4, x_1\}$
π_{15}	$\text{Bar}^{\mathcal{I}} = \{x_4, x_1, x_2\}$
π_{16}	$\text{Bar}^{\mathcal{I}} = \Delta^{\mathcal{I}}$

There are 8 conditions in this case.

$$\begin{aligned}&\text{Bar}^{\bar{\mathcal{I}}}(x_1) = 3/5\\&\text{Bar}^{\bar{\mathcal{I}}}(x_2) = 3/5\\&\text{Bar}^{\bar{\mathcal{I}}}(x_3) = 3/5\\&\text{Bar}^{\bar{\mathcal{I}}}(x_4) = 3/5\\&\text{Bar}^{\bar{\mathcal{I}}}(x_1 \vee x_2 \vee x_3) = 1\\&\text{Bar}^{\bar{\mathcal{I}}}(x_2 \vee x_3 \vee x_4) = 1\\&\text{Bar}^{\bar{\mathcal{I}}}(x_3 \vee x_4 \vee x_1) = 1\\&\text{Bar}^{\bar{\mathcal{I}}}(x_4 \vee x_1 \vee x_2) = 1\end{aligned}$$

Therefor the linear system looks like this.

$$A=\begin{bmatrix}1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\0&1&0&0&0&1&0&0&1&1&0&1&0&1&1&1\\0&0&1&0&0&1&1&0&0&0&1&1&1&0&1&1\\0&0&0&1&0&0&1&1&0&1&0&1&1&1&0&1\\0&0&0&0&1&0&0&1&1&0&1&0&1&1&1&1\\0&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1\\0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\0&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\0&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\end{bmatrix}$$

And

$$A\cdot\pi=\begin{bmatrix}1\\3/5\\3/5\\3/5\\3/5\\1\\1\\1\\1\end{bmatrix}$$

This system has several solutions, that fulfill the condition $\pi \geq 0$. The parameter must fulfill the condition $\frac{1}{10} \leq c \leq \frac{2}{15}$.

$$\pi=\begin{bmatrix}0\\0\\0\\0\\0\\c\\c\\c\\c\\c\\c\\2/5-3c\\2/5-3c\\2/5-3c\\2/5-3c\\6c-3/5\end{bmatrix}$$

In this case, the knowledge base is consistent. One can also give a range for the probability of each possible world, e.g. the probability, that all 4 meet in the bar tonight is between 0 and $1/5$.

The problem about this straight forward technique is that the amount of possible worlds is increasing exponentially.

And, of course, this is only possible with uncertainty and not with vagueness or imprecision. There are several possibilities to reduce the amount of calculations and to deal with vagueness. The following approach to add fuzziness to a description logic is described in [11].

Adding fuzziness to a classical description logic means that the interpretation changes. In description logics, individuals, concepts and roles get associated with elements of the domain directly. In fuzzy logics, every possible assertion with the domain gets an additional numeric value, which states the probability that the assertion exists. Concepts and roles are not subsets of the domain any more, but each element of the domain gets a probability to be a part of a specific concept or role. According to definition 3 one can define the fuzzy interpretation.

Definition 5: A fuzzy interpretation $\bar{\mathcal{I}} = \left\langle \Delta^{\bar{\mathcal{I}}}, \cdot^{\bar{\mathcal{I}}} \right\rangle$ over a signature (N_R, N_C, N_O) is a tuple, that consists of a domain $\Delta^{\bar{\mathcal{I}}} \neq \emptyset$ and a function with the following constraints. An individual $a \in N_O$ must be mapped to a function on the domain, i.e. $a^{\bar{\mathcal{I}}} : \Delta^{\bar{\mathcal{I}}} \to \{0,1\}$. A role $r \in \mathbf{R} = N_R$ must be mapped to a function on the Cartesian square of the domain, i.e. $r^{\bar{\mathcal{I}}} : \Delta^{\bar{\mathcal{I}}} \times \Delta^{\bar{\mathcal{I}}} \to [0,1]$. Again, according to definition 1, a concept can occur in different ways. Beside the fact, that each concept $C \in \mathbf{C}$ must be mapped to a function on the domain, i.e. $C^{\bar{\mathcal{I}}} : \Delta^{\mathcal{I}} \to [0,1]$, the composite concepts must fulfill the following rules, where $C, D \in \mathbf{C}$ are concepts, $r \in \mathbf{R}$ is a rule and $A = \{a_1, \ldots, a_n\} \subseteq N_O$ is a set of individuals. $\delta \in \Delta^{\bar{\mathcal{I}}}$ is an arbitrary element of the domain.

$$\top^{\bar{\mathcal{I}}}(\delta) = 1 \tag{2}$$

$$\bot^{\bar{\mathcal{I}}}(\delta) = 0 \tag{3}$$

$$A^{\bar{\mathcal{I}}}(\delta) = \begin{cases} 0 & a_1^{\bar{\mathcal{I}}}(\delta) = \ldots = a_n^{\bar{\mathcal{I}}}(\delta) = 0 \\ 1 & \text{otherwise} \end{cases} \tag{4}$$

$$(\neg C)^{\bar{\mathcal{I}}}(\delta) = \ominus C^{\bar{\mathcal{I}}}(\delta) \tag{5}$$

$$(C \sqcap D)^{\bar{\mathcal{I}}}(\delta) = C^{\bar{\mathcal{I}}}(\delta) \otimes D^{\bar{\mathcal{I}}}(\delta) \tag{6}$$

$$(C \sqcup D)^{\bar{\mathcal{I}}}(\delta) = C^{\bar{\mathcal{I}}}(\delta) \oplus D^{\bar{\mathcal{I}}}(\delta) \tag{7}$$

$$(\exists r.C)^{\bar{\mathcal{I}}}(\delta) = \mathbf{sup}_{\delta' \in \Delta^{\bar{\mathcal{I}}}} \left(r^{\bar{\mathcal{I}}}(\delta, \delta') \otimes C^{\bar{\mathcal{I}}}(\delta') \right) \tag{8}$$

$$(\forall r.C)^{\bar{\mathcal{I}}}(\delta) = \mathbf{inf}_{\delta' \in \Delta^{\bar{\mathcal{I}}}} \left(r^{\bar{\mathcal{I}}}(\delta, \delta') \rhd C^{\bar{\mathcal{I}}}(\delta') \right) \tag{9}$$

Definition 6: In a fuzzy interpretation $\bar{\mathcal{I}}$ holds a TBox $\mathcal{T}$, denoted by $\bar{\mathcal{I}} \models \mathcal{T}$, if and only if every GCI $(C \sqsubseteq D) \in \mathcal{T}$, where $C, D \in \mathbf{C}$ are concepts, fulfills $(C \sqsubseteq D)^{\bar{\mathcal{I}}} = \mathbf{inf}_{\delta \in \Delta^{\bar{\mathcal{I}}}} \left(C^{\bar{\mathcal{I}}}(\delta) \rhd D^{\bar{\mathcal{I}}}(\delta) \right)$.

In an interpretation $\mathcal{I}$ holds an ABox $\mathcal{A}$, denoted by $\mathcal{I} \models \mathcal{A}$, if and only if every individual assertion $C(a) \in \mathcal{A}$, where $a \in N_O$ is an individual name and $C \in \mathbf{C}$ is a concept, fulfills $a^{\mathcal{I}} \in C^{\mathcal{I}}$ and every individual assertion $r(a,b) \in \mathcal{A}$, where $a, b \in N_O$ are individual names and $r \in \mathbf{R}$ is a role, fulfills $\left\langle a^{\mathcal{I}}, b^{\mathcal{I}} \right\rangle \in r^{\mathcal{I}}$.

In an interpretation $\mathcal{I}$ holds a knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, denoted by $\mathcal{I} \models \mathcal{K}$, if and only if $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$.

A knowledge base $\mathcal{K}$ is called consistent, if and only if there exists an interpretation $\mathcal{I}$, such that it holds the knowledge base, i.e. $\mathcal{I} \models \mathcal{K}$. It is called inconsistent otherwise.

The three binary operators $\otimes$, $\oplus$, $\rhd$ and the unary operator $\ominus$ have to be chosen carefully. The next section shows that there shouldn't be an arbitrary choice. A useful choice would be in a way that the fuzzy logic is an extension of the non-fuzzy logic, i.e. that the former workings are still valid.

Nevertheless, whichever formula one choses for the definition of these operators, since the numeric value doesn't remind the context where it is extracted from, the behavior of the operators is unexpected in several cases, i.e. some argument forms in propositional logic are not fulfilled any more. For example, it is impossible that $\ominus \ominus x = x$ (Double Negation), $x \leq y \implies \ominus y \leq \ominus x$ (monotonically decreasing Negation), $x \oplus x = x$ (Tautology) and $\ominus x \oplus x = 1$ (Law of Excluded Middle) occur simultaneously.

Proof: Assuming that every $x, y \in [0,1]$ holds $\ominus \ominus x = x$, $x \leq y \implies \ominus y \leq \ominus x$, $x \oplus x = x$ and $\ominus x \oplus x = 1$. The following paragraph shows that $\ominus$ is injective and surjective on $[0,1]$. Together with the monotonicity, this implies that it has a fixed point. This fixed point is used to show that Tautology and the Law of Excluded Middle are impossible at the same time.

Since $\ominus \ominus x = x$, all negated values must be different. If there are $a, b \in [0,1]$ with $\ominus a = \ominus b$, then $a = \ominus \ominus a = \ominus(\ominus a) = \ominus(\ominus b) = \ominus \ominus b = b$, i.e. for every $a, b \in [0,1]$: $a \neq b$ implies $\ominus a \neq \ominus b$. And for every $c \in [0,1]$, you can find an $a \in [0,1]$, such that $\ominus a = c$. This is easy to find by setting $a := \ominus c$. Then $\ominus a = \ominus(\ominus c) = \ominus\ominus c = c$. All together, the operator $\ominus$ is a bijection.

Because of the monotonicity of $\ominus$, there is exactly one $0 < z < 1$, such that $\ominus z = z$. Let $A = \{x \in [0,1] : x < \ominus x\}$ and $B = \{x \in [0,1] : x > \ominus x\}$ be a partition of $[0,1]$. For every $a \in A$ and $b \in B$, it is $a < b$. Otherwise, if there would exist $a \in A$ and $b \in B$ with $a \leq b$, then $\ominus a > a \leq b > \ominus b$, which is a contradiction to the monotonicity of $\ominus$. Let $z := \mathbf{sup}(A)$ be the supremum of A, i.e. z is the smallest possible $z \geq a$ for any $a \in A$. One can easily see, because of the Double Negation, for every $a \in A$ and $b \in B$, it is $\ominus a \in B$ and $\ominus b \in A$. Assuming, that $z \in A$. Then $\ominus z > z$ with $\ominus z \in B$ and therefor exists another $z' \in B$ with $\ominus z > z' > z$ and $\ominus z' \in A$. On the other hand $\ominus z > z' = \ominus \ominus z'$ and hence $z < \ominus z'$, which means that z was not the supremum of A. This showed, that $z \notin A$ and in the same way one can show, that $z = \mathbf{inf}(B)$ and $z \notin B$.

This fixed point z can not fulfill both, $z \oplus z = z$ and $\ominus z \oplus z = 1$, since $\ominus z \oplus z = z \oplus z$ and $z \neq 1$.

There are several other forms that contradict each other. For example, Modus Ponens, i.e. $(a \rhd b) \otimes a = b$, should not be fulfilled by a useful fuzzy logic.

V. TRANSFORMATION

This section describes the process of transforming a knowledge base in description logic to a knowledge base with the management of uncertainty. The starting point is a description logic with ABox and TBox, but without probabilistic values. One can simply include these boxes into a fuzzy logic. A further approach is the enrichment of data with uncertainty, i.e. the estimation of probability for several kinds of knowledge that were already known before or that were unknown before. To come back from fuzzy logic to description logic, one has to remove the uncertainty and include only knowledge that has at least a certain degree of certainty within the fuzzy knowledge base.

To do an inclusion from the description logic with an interpretation $\mathcal{I}$ into a fuzzy logic, one can construct the fuzzy interpretation $\bar{\mathcal{I}}$ in the following way, where $a \in N_O$ is an individual name, $C \in N_C$ is a concept name, $r \in N_R$ is a role name and $\delta, \delta' \in \Delta^{\mathcal{I}} = \Delta^{\bar{\mathcal{I}}}$.

$$\delta = a^{\mathcal{I}} \iff a^{\bar{\mathcal{I}}}(\delta) = 1 \tag{10}$$
$$\delta \neq a^{\mathcal{I}} \iff a^{\bar{\mathcal{I}}}(\delta) = 0 \tag{11}$$
$$\delta \in C^{\mathcal{I}} \iff C^{\bar{\mathcal{I}}}(\delta) = 1 \tag{12}$$
$$\delta \notin C^{\mathcal{I}} \iff C^{\bar{\mathcal{I}}}(\delta) = 0 \tag{13}$$
$$\langle \delta, \delta' \rangle \in r^{\mathcal{I}} \iff r^{\bar{\mathcal{I}}}(\delta, \delta') = 1 \tag{14}$$
$$\langle \delta, \delta' \rangle \notin r^{\mathcal{I}} \iff r^{\bar{\mathcal{I}}}(\delta, \delta') = 0 \tag{15}$$

To show, that this construction also works with composite concepts, let $C \in \mathbf{C}$ be an arbitrary concept. One must now show that in every case of composition the result in the fuzzy logic (Definition 5) is the same result in the non-fuzzy logic (Definition 3), i.e. for all $\delta \in \Delta^{\mathcal{I}} = \Delta^{\bar{\mathcal{I}}}$ is $\delta \in C^{\mathcal{I}} \implies C^{\bar{\mathcal{I}}}(\delta) = 1$ and $\delta \notin C^{\mathcal{I}} \implies C^{\bar{\mathcal{I}}}(\delta) = 0$.

(1) $C \in N_R$:

$$\delta \in C^{\mathcal{I}} \overset{(12)}{\implies} C^{\bar{\mathcal{I}}}(\delta) = 1$$

$$\delta \notin C^{\mathcal{I}} \overset{(13)}{\implies} C^{\bar{\mathcal{I}}}(\delta) = 0$$

(2) $\top$:

$$\delta \in \Delta^{\mathcal{I}} \overset{(2)}{\implies} \delta \in \top^{\mathcal{I}}$$
$$\delta \in \Delta^{\bar{\mathcal{I}}} \overset{(2)}{\implies} \top^{\bar{\mathcal{I}}}(\delta) = 1$$

(3) $\bot$:

$$\delta \in \Delta^{\mathcal{I}} \overset{(3)}{\implies} \delta \notin \bot^{\mathcal{I}}$$
$$\delta \in \Delta^{\bar{\mathcal{I}}} \overset{(3)}{\implies} \bot^{\bar{\mathcal{I}}}(\delta) = 0$$

(4) $A = \{a_1, \dots a_n\} \subseteq N_O$:

$$\delta \in A^{\mathcal{I}} \overset{(4)}{\implies} \delta \in \{a_1^{\mathcal{I}}, \dots a_n^{\mathcal{I}}\}$$
$$\overset{(10)}{\implies} \exists a \in \{a_1^{\mathcal{I}}, \dots a_n^{\mathcal{I}}\} : a^{\bar{\mathcal{I}}}(\delta) = 1 \neq 0 \overset{(4)}{\implies} A^{\bar{\mathcal{I}}}(\delta) = 1$$

$$\delta \notin A^{\mathcal{I}} \overset{(4)}{\implies} \delta \notin \{a_1^{\mathcal{I}}, \dots a_n^{\mathcal{I}}\}$$
$$\overset{(11)}{\implies} \forall a \in \{a_1^{\mathcal{I}}, \dots a_n^{\mathcal{I}}\} : a^{\bar{\mathcal{I}}}(\delta) = 0 \overset{(4)}{\implies} A^{\bar{\mathcal{I}}}(\delta) = 0$$

(5) $\neg C : C \in \mathbf{C}$:

$$\delta \in (\neg C)^{\mathcal{I}} \overset{(5)}{\implies} \delta \notin C^{\mathcal{I}} \overset{(13)}{\implies} C^{\bar{\mathcal{I}}}(\delta) = 0 \overset{(5)}{\implies} (\neg C)^{\bar{\mathcal{I}}}(\delta) = \ominus 0$$

$$\delta \notin (\neg C)^{\mathcal{I}} \overset{(5)}{\implies} \delta \in C^{\mathcal{I}} \overset{(12)}{\implies} C^{\bar{\mathcal{I}}}(\delta) = 1 \overset{(5)}{\implies} (\neg C)^{\bar{\mathcal{I}}}(\delta) = \ominus 1$$

Conclusion: $\ominus 0 = 1$ and $\ominus 1 = 0$.

(6) $C \sqcap D : C, D \in \mathbf{C}$:

$$\delta \in (C \sqcap D)^{\mathcal{I}} \overset{(6)}{\implies} \delta \in C^{\mathcal{I}} \cap D^{\mathcal{I}} \implies \delta \in C^{\mathcal{I}} \wedge \delta \in D^{\mathcal{I}}$$
$$\overset{(12)}{\implies} C^{\bar{\mathcal{I}}}(\delta) = 1 \wedge D^{\bar{\mathcal{I}}}(\delta) = 1 \overset{(19)}{\implies} (C \sqcap D)^{\bar{\mathcal{I}}}(\delta) = 1 \otimes 1$$

$$\delta \notin (C \sqcap D)^{\mathcal{I}} \overset{(6)}{\implies} \delta \notin C^{\mathcal{I}} \cap D^{\mathcal{I}} \implies \delta \notin C^{\mathcal{I}} \vee \delta \notin D^{\mathcal{I}}$$
$$\overset{(13)}{\implies} C^{\bar{\mathcal{I}}}(\delta) = 0 \vee D^{\bar{\mathcal{I}}}(\delta) = 0 \overset{(6), wlog}{\implies} (C \sqcap D)^{\bar{\mathcal{I}}}(\delta) = C^{\bar{\mathcal{I}}}(\delta) \otimes 0$$

Conclusion: $1 \otimes 1 = 1$ and $0 \otimes x = 0$ for any $x \in [0, 1]$.

(7) $C \sqcup D : C, D \in \mathbf{C}$:

$$\delta \in (C \sqcup D)^{\bar{\mathcal{I}}} \overset{(7)}{\implies} \delta \in C^{\mathcal{I}} \cup D^{\mathcal{I}} \implies \delta \in C^{\mathcal{I}} \vee \delta \in D^{\mathcal{I}}$$
$$\overset{(12)}{\implies} C^{\bar{\mathcal{I}}}(\delta) = 1 \vee D^{\bar{\mathcal{I}}}(\delta) = 1 \overset{(7), wlog}{\implies} (C \sqcup D)^{\bar{\mathcal{I}}}(\delta) = C^{\bar{\mathcal{I}}}(\delta) \oplus 1$$

$$\delta \notin (C \sqcup D)^{\mathcal{I}} \overset{(7)}{\implies} \delta \notin C^{\mathcal{I}} \cup D^{\mathcal{I}} \implies \delta \notin C^{\mathcal{I}} \wedge \delta \notin D^{\mathcal{I}}$$
$$\overset{(13)}{\implies} C^{\bar{\mathcal{I}}}(\delta) = 0 \wedge D^{\bar{\mathcal{I}}}(\delta) = 0 \overset{(7)}{\implies} (C \sqcup D)^{\bar{\mathcal{I}}}(\delta) = 0 \oplus 0$$

Conclusion: $0 \oplus 0 = 0$ and $1 \oplus x = 1$ for any $x \in [0, 1]$.

(8)$\exists r.C : C \in \mathbf{C}, r \in \mathbf{R}$:

$$\delta \in (\exists r.C)^{\mathcal{I}} \overset{(8)}{\implies} \exists \delta' \in \Delta^{\mathcal{I}} : \delta' \in C^{\mathcal{I}} \wedge \langle \delta, \delta' \rangle \in r^{\mathcal{I}}$$
$$\overset{(12),(14)}{\implies} \exists \delta' \in \Delta^{\bar{\mathcal{I}}} : C^{\bar{\mathcal{I}}}(\delta') = 1 \wedge r^{\bar{\mathcal{I}}}(\delta, \delta') = 1$$
$$\overset{(8)}{\implies} (\exists r.C)^{\mathcal{I}}(\delta) = \mathbf{sup}_{\delta' \in \Delta^{\mathcal{I}}} \left(r^{\mathcal{I}}(\delta, \delta') \otimes C^{\mathcal{I}}(\delta') \right) \overset{1 \otimes 1 = 1}{=} 1$$

$$\delta \notin B^{\bar{\mathcal{I}}} \overset{(10)}{\implies} \forall \delta' \in \Delta^{\mathcal{I}} : \delta' \notin C^{\bar{\mathcal{I}}} \vee \langle \delta, \delta' \rangle \notin r^{\bar{\mathcal{I}}}$$
$$\overset{(24)}{\implies} \forall \delta' \in \Delta^{\mathcal{I}} : C^{\mathcal{I}}(\delta') = 0 \vee r^{\mathcal{I}}(\delta, \delta') = 0$$
$$\overset{(21)}{\implies} B^{\mathcal{I}}(\delta) = \mathbf{sup}_{\delta' \in \Delta^{\mathcal{I}}} \left(r^{\mathcal{I}}(\delta, \delta') \otimes C^{\mathcal{I}}(\delta') \right) \overset{x \otimes 0 = 0 \otimes x = 0}{=} 0$$

(9)$\forall r.C : C \in \mathbf{C}, r \in \mathbf{R}$:

$$\delta \in B^{\bar{\mathcal{I}}} \overset{(11)}{\Longrightarrow} \forall \delta' \in \Delta^{\mathcal{I}} : \delta' \in C^{\bar{\mathcal{I}}} \vee \langle \delta, \delta' \rangle \notin r^{\bar{\mathcal{I}}}$$
$$\overset{(23),(24)}{\Longrightarrow} \forall \delta' \in \Delta^{\mathcal{I}} : C^{\mathcal{I}}(\delta') = 1 \vee r^{\mathcal{I}}(\delta, \delta') = 0$$
$$\overset{(22)}{\Longrightarrow} B^{\mathcal{I}}(\delta) = \mathbf{inf}_{\delta' \in \Delta^{\mathcal{I}}} \left(r^{\mathcal{I}}(\delta, \delta') \triangleright C^{\mathcal{I}}(\delta') \right) \overset{?}{=} 1$$

$$\delta \notin B^{\bar{\mathcal{I}}} \overset{(11)}{\Longrightarrow} \exists \delta' \in \Delta^{\mathcal{I}} : \delta' \notin C^{\bar{\mathcal{I}}} \wedge \langle \delta, \delta' \rangle \in r^{\bar{\mathcal{I}}}$$
$$\overset{(23),(24)}{\Longrightarrow} \exists \delta' \in \Delta^{\mathcal{I}} : C^{\mathcal{I}}(\delta') = 0 \wedge r^{\mathcal{I}}(\delta, \delta') = 1$$
$$\overset{(22)}{\Longrightarrow} B^{\mathcal{I}}(\delta) = \mathbf{inf}_{\delta' \in \Delta^{\mathcal{I}}} \left(r^{\mathcal{I}}(\delta, \delta') \triangleright C^{\mathcal{I}}(\delta') \right) \overset{?}{=} 0$$

Conclusion: $1 \triangleright 0 = 0$ and $x \triangleright 1 = 0 \triangleright x = 1$ for any $x \in [0, 1]$.

The choice of the four operators should fulfill these concluded conditions. Otherwise the transformation will be inconsistent, i.e. the transformation from description logic to fuzzy logic and back could result in a different knowledge base. Table 1 in [11] shows a variety of fuzzy logics, that satisfy the constraints.

Lukasiewicz Logic (LL)

$$\begin{aligned} a \otimes b &= \max(a + b - 1, 0) \\ a \oplus b &= \min(a + b, 1) \\ a \triangleright b &= \min(1 - a + b, 1) \\ \ominus a &= 1 - a \end{aligned}$$

Goedel Logic (GL)

$$\begin{aligned} a \otimes b &= \min(a, b) \\ a \oplus b &= \max(a, b) \\ a \triangleright b &= \begin{cases} 1 & a \leq b \\ b & a > b \end{cases} \\ \ominus a &= \begin{cases} 1 & a = 0 \\ 0 & a > 0 \end{cases} \end{aligned}$$

Product Logic (PL)

$$\begin{aligned} a \otimes b &= a \cdot b \\ a \oplus b &= a + b - a \cdot b \\ a \triangleright b &= \min(1, b/a) \\ \ominus a &= \begin{cases} 1 & a = 0 \\ 0 & a > 0 \end{cases} \end{aligned}$$

Zadeh Logic (ZL)

$$\begin{aligned} a \otimes b &= \min(a, b) \\ a \oplus b &= \max(a, b) \\ a \triangleright b &= \max(1 - a, b) \\ \ominus a &= 1 - a \end{aligned}$$

VI. Conclusion

This paper described the management of knowledge with and without the management of uncertainty. Description logics have been described in section III and fuzzy logics in section IV. It is easy to get from description logic to fuzzy logic (cf section V).

For the other way to get knowledge from a fuzzy logic to a description logic, it is impossible to get the same result when going back to fuzzy logic again. This is obvious, since the concept of every fuzzy logic is build on top of a description logic, i.e. a fuzzy logic is more powerful.

The following paragraph describes the reduction with a threshold $\tau \in [0, 1]$ to get from a fuzzy logic to a description logic. To do a reduction from the fuzzy logic with an interpretation $\bar{\mathcal{I}}$ into a description logic, one can construct the non-fuzzy interpretation $\mathcal{I}$ in the following way, where $a \in N_O$ is an individual name, $C \in N_C$ is a concept name, $r \in N_R$ is a role name and $\delta, \delta' \in \Delta^{\mathcal{I}} = \Delta^{\bar{\mathcal{I}}}$.

$$\delta = a^{\mathcal{I}} \iff a^{\bar{\mathcal{I}}}(\delta) = 1 \tag{10}$$
$$\delta \neq a^{\mathcal{I}} \iff a^{\bar{\mathcal{I}}}(\delta) = 0 \tag{11}$$
$$\delta \in C^{\mathcal{I}} \iff C^{\bar{\mathcal{I}}}(\delta) \geq \tau \tag{12}$$
$$\delta \notin C^{\mathcal{I}} \iff C^{\bar{\mathcal{I}}}(\delta) < \tau \tag{13}$$
$$\langle \delta, \delta' \rangle \in r^{\mathcal{I}} \iff r^{\bar{\mathcal{I}}}(\delta, \delta') \geq \tau \tag{14}$$
$$\langle \delta, \delta' \rangle \notin r^{\mathcal{I}} \iff r^{\bar{\mathcal{I}}}(\delta, \delta') < \tau \tag{15}$$

To show, that this construction also works with composite concepts, let $C \in \mathbf{C}$ be an arbitrary concept. One must now show that in every case of composition the result in the description logic (Definition 5) is the same result in the fuzzy logic (Definition 3), i.e. for all $\delta \in \Delta^{\mathcal{I}} = \Delta^{\bar{\mathcal{I}}}$ is $C^{\bar{\mathcal{I}}}(\delta) \geq \tau \implies \delta \in C^{\mathcal{I}}$ and $C^{\bar{\mathcal{I}}}(\delta) < \tau \implies \delta \notin C^{\mathcal{I}}$.

Since it's already proven in section IV that Double Negation, monotonically decreasing Negation, Tautology and Law of Excluded Middle cannot occur simultaneously, this can not work for an arbitrary τ, no matter how the operators are chosen. The open question is, if there is at least another possibility to construct a consistent description logic for the non-trivial case, where $\tau \neq 0$ and $\tau \neq 1$.

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